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2006 J. Phys. A: Math. Gen. 39 157

(http://iopscience.iop.org/0305-4470/39/1/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 157–162 doi:10.1088/0305-4470/39/1/011

Quantum-mechanical time evolution and uniform forces

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Received 19 August 2005, in final form 27 October 2005
Published 7 December 2005
Online at stacks.iop.org/JPhysA/39/157

Abstract

The time evolution of an arbitrary one-dimensional initial quantum state, subject to a time-independent, spatially uniform classical force, is shown to be that of a free-particle state, plus an overall motion arising from the classical force. Bohmian mechanics is then used to extend this result to an arbitrarily time-dependent, uniform force in three dimensions. The resulting solution is completed and confirmed using conventional quantum mechanics.

PACS number: 03.65.−w

1. Introduction

The time evolution of a quantum system subject to a spatially uniform, time-dependent force has attracted considerable interest of late. The exact propagator for this system has long been known [1], as have a set of exact solutions (the Volkov solutions) [2–4]. Recent work [4–11] focuses further on exact solutions and their properties. Here, in contrast, we obtain the general result that the uniform force case may be effectively reduced to the free-particle case.

Section 2 employs analytical methods to examine the time evolution of an arbitrary one-dimensional initial state subject to a spatially uniform, time-independent force. Section 3 introduces Bohmian mechanics, which is then used in section 4 to examine the evolution of the probability distribution for an arbitrary three-dimensional initial state subject to a uniform, arbitrarily time-dependent force. Section 5 extends the results of section 4 to quantum states. In section 6 our results are completed and verified. Section 7 sets our results in the context of classical expectations.

2. Time-independent forces

In 1930, deBroglie [12, 13] showed that the time evolution of a one-dimensional Gaussian subject to a spatially uniform, time-independent force is precisely that of a free-particle

Gaussian, apart from an overall translation and momentum boost identical to what the uniform force would have imparted to a classical system. Andrews [14] has recently shown that this also holds when the uniform force exhibits an arbitrary time dependence. These results, although exact, rely on sophisticated mathematical tools, and are limited to Gaussians.

deBroglie's result for a Gaussian may be extended to arbitrary initial states with a simple analytical approach. Take the classical potential to be $V = -F_{\mathrm{CL}}x$, with F_{CL} constant. The momentum-space Schrödinger equation is

$$\frac{\partial\phi(p,t)}{\partial t} + F_{\mathrm{CL}}\frac{\partial\phi(p,t)}{\partial p} + \frac{\mathrm{i}p^2}{2m\hbar}\phi(p,t) = 0. \tag{1}$$

The method of characteristics leads to the solution

$$\phi(p,t) = g(p - F_{\mathrm{CL}}t)\exp\left\{\frac{-\mathrm{i}}{2m\hbar}\int_0^t (p - F_{\mathrm{CL}}t')^2\,\mathrm{d}t'\right\}, \tag{2}$$

where $g(p)$ is an arbitrary initial momentum distribution. The corresponding free-particle equation and its solution are

$$\frac{\partial\phi_f(p,t)}{\partial t} + \frac{\mathrm{i}p^2}{2m\hbar}\phi_f(p,t) = 0, \tag{3}$$

$$\phi_f(p,t) = g(p)\exp\left(\frac{-\mathrm{i}}{2m\hbar}p^2 t\right). \tag{4}$$

The position-space wavefunctions are the Fourier transforms of (2) and (4):

$$\Psi(x,t) = C\int_{-\infty}^{\infty} g(p - F_{\mathrm{CL}}t)\exp\left(\frac{\mathrm{i}px}{\hbar}\right)\exp\left\{\frac{-\mathrm{i}}{2m\hbar}\int_0^t (p - F_{\mathrm{CL}}t')^2\,\mathrm{d}t'\right\}\mathrm{d}p, \tag{5}$$

$$\Psi_f(x,t) = C\int_{-\infty}^{\infty} g(p)\exp\left\{\mathrm{i}\left(\frac{px}{\hbar} - \frac{p^2 t}{2m\hbar}\right)\right\}\mathrm{d}p. \tag{6}$$

In the time interval $[0,t]$, the classical displacement and momentum boost due to F_{CL} are $\Delta x_{\mathrm{CL}} = F_{\mathrm{CL}}t^2/2m$ and $\Delta p_{\mathrm{CL}} = F_{\mathrm{CL}}t$, respectively. Using the variable substitution $p \to y + \Delta p_{\mathrm{CL}}$, then, (5) becomes

$$\Psi(x,t) = C\exp\left(\frac{-\mathrm{i}F_{\mathrm{CL}}^2 t^3}{6m\hbar}\right)\exp\left(\frac{\mathrm{i}x\Delta p_{\mathrm{CL}}}{\hbar}\right)\int_{-\infty}^{\infty} g(y)\exp\left\{\mathrm{i}\left(\frac{y(x-\Delta x_{\mathrm{CL}})}{\hbar} - \frac{y^2 t}{2m\hbar}\right)\right\}\mathrm{d}y. \tag{7}$$

Comparing (6) and (7) we obtain, up to an unimportant overall phase factor,

$$\Psi(x,t) = \exp\left(\frac{\mathrm{i}x\Delta p_{\mathrm{CL}}}{\hbar}\right)\Psi_f\left(x - \Delta x_{\mathrm{CL}}, t\right). \tag{8}$$

Thus, the time evolved free and 'forced' states are identical, except that the classical motion is superimposed on the latter.

3. Bohmian mechanics and forces

In sections 4 and 5, we will use Bohmian mechanics to extend our results. Detailed treatments of Bohmian mechanics are readily available [13, 15–17]—here we present only those features necessary for our discussion. We start with the time-dependent Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\Psi(\mathbf{x},t) + V(\mathbf{x},t)\Psi(\mathbf{x},t). \tag{9}$$

We may write $\Psi(\mathbf{x}, t)$ in polar form, $\Psi(\mathbf{x}, t) = R(\mathbf{x}, t) \exp\{iS(\mathbf{x}, t)/\hbar\}$, where the modulus, R, and the phase, $S/\hbar$, are real functions. Substituting this form into (9), taking derivatives, and separating into real and imaginary parts, we obtain two coupled equations:

$$-\frac{\partial S}{\partial t} = \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\left(\frac{\nabla^2 R}{R}\right) \tag{10}$$

$$\frac{\partial(R^2)}{\partial t} = -\nabla \cdot \left(\left(\frac{R^2}{m}\right)\nabla S\right). \tag{11}$$

Note that $R^2 = |\Psi|^2$; that is, R^2 is simply the position probability distribution. Equation (11) may be cast into a continuity equation for the probability density, while (10), except for its last term, resembles the classical Hamilton–Jacobi equation:

$$-\frac{\partial S_{\mathrm{CL}}}{\partial t} = \frac{(\nabla S_{\mathrm{CL}})^2}{2m} + V, \tag{12}$$

where S_{CL} is *Hamilton's principal function* [18]. In Bohmian mechanics we regard (10) as a modified Hamilton–Jacobi equation for the quantum state, which now describes an ensemble of possible trajectories, with well-defined positions and momenta, corresponding to an ensemble of possible initial conditions of a *real* quantum particle. Because the position probability distribution for this real particle is given by $|\Psi(\mathbf{x}, t)|^2$, we obtain the usual quantum-mechanical predictions for position measurements.

The last term in (10) may be regarded as a *quantum potential energy*, Q, with which we may associate a *quantum force*, $\mathbf{F}_Q$:

$$Q = -\frac{\hbar^2}{2m}\left(\frac{\nabla^2 R}{R}\right) \qquad \mathbf{F}_Q = -\nabla Q. \tag{13}$$

Note that both Q and $\mathbf{F}_Q$ generally differ for different Bohmian trajectories in a state. The dynamics is determined by the total (classical plus quantum) force $\mathbf{F}$:

$$\mathbf{F} = -\nabla(V + Q). \tag{14}$$

Although the actual particle occupies only one Bohmian trajectory (although we do not know which one), $\mathbf{F}$ acts on each trajectory *as though* it were occupied. Thus, the state both determines the position probability distribution and influences the dynamics.

Solutions to the Schrödinger equation describe quantum states and their time evolution, incorporating within them quantum effects, the effects of the classical potential and the overall motion of the state. In Bohmian mechanics we may disentangle these effects. From (13), Q and $\mathbf{F}_Q$ depend only on the modulus, R, and are independent of the phase (but only in a limited sense; cf. section 4). Thus, the quantum effects arise only from the 'shape' of the state. From (14), we see that the total force $\mathbf{F}$ is separable into classical and quantum contributions. This feature may be of considerable benefit, even though the interplay between the two contributions may in general be very complicated.

Bohmian mechanics may also be formulated without reference to forces. In the Hamilton–Jacobi formulation of classical mechanics, one obtains a relation between S_{CL} and the canonical momentum, $\mathbf{p}_{\mathrm{CL}}$ [18]. Analogous to this is a relation between S and the Bohmian momentum $\mathbf{p}$:

$$\mathbf{p}_{\mathrm{CL}} = \nabla S_{\mathrm{CL}} \quad \Longleftrightarrow \quad \mathbf{p} = \nabla S. \tag{15}$$

If we take $\mathbf{p} = \nabla S$ as our dynamical law, there need be no mention of forces. In actual calculations using this approach, we must first obtain the wavefunction using standard quantum-mechanical methods, and then use $\mathbf{p} = \nabla S$ to obtain the Bohmian trajectories. Since this formulation of Bohmian mechanics is less wedded to classical mechanics, invoking neither forces nor Newton's second law, some researchers see it as preferable to the quantum force formulation.

4. Time-dependent forces

The quantum force formulation of Bohmian mechanics provides a new means to time evolve the quantum state, by applying Newton's second law to the separable quantum and classical forces. The result of section 2 can be easily obtained, and extended, using this approach.

We first revisit the time evolution of an arbitrary initial state, denoted by $\Psi(\mathbf{x}, 0)$, in the spatially linear, time-independent potential $V = -F_{\mathrm{CL}}x$. From (14), we have

$$\mathbf{F} = -\nabla(V + Q) = \nabla\left\{F_{\mathrm{CL}}x + \frac{\hbar^2}{2m}\left(\frac{\nabla^2 R}{R}\right)\right\}$$
$$= F_{\mathrm{CL}}\hat{x} + \frac{\hbar^2}{2m}\nabla\left(\frac{\nabla^2 R}{R}\right). \tag{16}$$

One cannot conclude on the basis of separability alone that F_{CL} cannot influence $\mathbf{F}_Q$. In general, F_{CL} will alter the momentum distribution (and thus, by (15), the phase). This, over time, may alter the modulus R, so $\mathbf{F}_Q$ will differ from its value were $F_{\mathrm{CL}} = 0$. This illustrates two effects referred to earlier: the complicated interplay between classical and quantum forces, and the limited sense in which Q and $\mathbf{F}_Q$ are independent of the phase.

In the case at hand, however, $F_{\mathrm{CL}}\hat{x}$ is spatially uniform. It thus imparts an identical momentum boost to each Bohmian trajectory. As a result, $F_{\mathrm{CL}}\hat{x}$ will translate the modulus R, but it cannot alter the momenta of the Bohmian trajectories relative to each other, nor can it alter R's shape. Because F_{CL} cannot alter R's shape, $\mathbf{F}_Q$ must remain identical for corresponding trajectories in the free and forced states. Thus, the two states' moduli must time evolve identically, apart from the overall translation and boost arising from $F_{\mathrm{CL}}\hat{x}$.

Now consider a classical force $\mathbf{F}_{\mathrm{CL}}(t) = F_{\mathrm{CL}}(t)\hat{e}(t)$, where both $F_{\mathrm{CL}}(t)$ and the unit vector $\hat{e}(t)$ exhibit arbitrary time dependences. That is, $\mathbf{F}_{\mathrm{CL}}(t)$ is spatially uniform, but arbitrarily time dependent in both magnitude and direction.

Our results for the time-independent force, $F_{\mathrm{CL}}\hat{x}$, rely only on the spatial uniformity of $F_{\mathrm{CL}}\hat{x}$ and the separability of $F_{\mathrm{CL}}\hat{x}$ and $\mathbf{F}_Q$. But $\mathbf{F}_{\mathrm{CL}}(t)$ and $\mathbf{F}_Q$ are also separable, and $\mathbf{F}_{\mathrm{CL}}(t)$ remains spatially uniform for all t. Thus, $\mathbf{F}_{\mathrm{CL}}(t)$ affects each trajectory identically, as did $F_{\mathrm{CL}}\hat{x}$. As for $F_{\mathrm{CL}}\hat{x}$, then, $\mathbf{F}_{\mathrm{CL}}(t)$ can translate the modulus R, but it cannot alter its shape. Thus, the free and forced moduli must again time evolve identically, except for the overall translation and momentum boost due to $\mathbf{F}_{\mathrm{CL}}(t)$. Then we may write

$$P(\mathbf{x}, t) = P_f(\mathbf{x} - \Delta\mathbf{x}_{\mathrm{CL}}(t), t), \tag{17}$$

where P and P_f are the forced and free position probability distributions, respectively. Here

$$\Delta\mathbf{x}_{\mathrm{CL}}(t) = \int_0^t \frac{\Delta\mathbf{p}_{\mathrm{CL}}(t')}{m}\,\mathrm{d}t', \tag{18}$$

$$\Delta\mathbf{p}_{\mathrm{CL}}(t) = \int_0^t \mathbf{F}_{\mathrm{CL}}(t')\,\mathrm{d}t'. \tag{19}$$

That is, $\Delta\mathbf{x}_{\mathrm{CL}}(t)$ and $\Delta\mathbf{p}_{\mathrm{CL}}(t)$ are the classical displacement and momentum change, respectively, due to $\mathbf{F}_{\mathrm{CL}}(t)$.

5. Quantum states

Section 4 focused on the time evolution of Bohmian ensembles and probability distributions, rather than that of the quantum state *per se*. Our results thus inform us of the behaviour of the modulus R, but what of the phase? That is, does the entire state—modulus *and* phase—time evolve as the corresponding free-particle state, Ψ_f, plus classical effects due to $\mathbf{F}_{\mathrm{CL}}(t)$?

A uniform classical force imparts a corresponding uniform boost to each trajectory. Thus, the Bohmian momentum distribution of the forced ensemble is identical to that of the free ensemble, apart from $\Delta\mathbf{p}_{\mathrm{CL}}(t)$, the boost due to $\mathbf{F}_{\mathrm{CL}}(t)$. But Bohmian momenta are determined solely by the phase, through $\mathbf{p} = \nabla S$. Thus, the two states' phases must correspond apart from the boost (and possibly an unimportant additive term, denoted by $S_0/\hbar$, which may depend on t, but not on $\mathbf{x}$). We may now construct an analogue to (17) in terms of quantum states:

$$\Psi(\mathbf{x}, t) = \Psi_f(\mathbf{x} - \Delta\mathbf{x}_{\mathrm{CL}}(t), t)\exp(\mathrm{i}\Delta\mathbf{p}_{\mathrm{CL}}(t)\cdot\mathbf{x}/\hbar)\exp(\mathrm{i}S_0/\hbar). \tag{20}$$

Although we have not proven (20), we regard it as well grounded in the preceding qualitative argument. (Note that (20) and (8) are analogues, the latter being restricted to one dimension and time-independent forces.)

Our use of Bohmian mechanics made the generalization to three-dimensional, time-dependent forces simple. Our discussion has been somewhat mathematical in nature. But once one grasps the separability of the classical and quantum forces, and sees how these forces act on an ensemble of Bohmian trajectories, the physics becomes almost obvious, and easily transparent to a truly conceptual understanding.

6. Confirmation

For our spatially uniform, arbitrarily time-dependent force, (9) becomes

$$\mathrm{i}\hbar\frac{\partial\Psi(\mathbf{x}, t)}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\Psi(\mathbf{x}, t) - \{\mathbf{F}(t)\cdot\mathbf{x}\}\Psi(\mathbf{x}, t). \tag{21}$$

To confirm our Bohmian analysis, we substitute (20) into (21). This leads to

$$\mathrm{d}S_0 = -\frac{\Delta\mathbf{p}^2_{\mathrm{CL}}(t)}{2m}\,\mathrm{d}t, \tag{22}$$

where $\Delta\mathbf{p}^2_{\mathrm{CL}}(t) = \Delta\mathbf{p}_{\mathrm{CL}}(t)\cdot\Delta\mathbf{p}_{\mathrm{CL}}(t)$. Solving for S_0, we obtain the explicit form of the state:

$$\Psi(\mathbf{x}, t) = \Psi_f(\mathbf{x} - \Delta\mathbf{x}_{\mathrm{CL}}(t), t)\exp(\mathrm{i}\Delta\mathbf{p}_{\mathrm{CL}}(t)\cdot\mathbf{x}/\hbar)\exp\left(\frac{-\mathrm{i}}{2m\hbar}\int_0^t \Delta\mathbf{p}^2_{\mathrm{CL}}(t')\,\mathrm{d}t'\right). \tag{23}$$

We have thus both determined the phase and confirmed our Bohmian analysis. In sum, the only effects of a three-dimensional, arbitrarily time-dependent, spatially uniform classical force are to translate and boost the corresponding free-particle quantum state as it would a classical particle.

We remark that some recent results are in fact special cases of our general result. Gaussian wave-packet solutions for the free particle are textbook fare [19]; Luan and Tang [10] find such solutions exist for the time-dependent linear potential. Dunkel and Trigger consider corresponding initial minimum-uncertainty Gaussians for both the free-particle case and a sinusoidally time-dependent linear potential; they find that the position/momentum uncertainty products and the joint entropies are identical [11].

7. Classical perspective

Classical physics provides further insight into our results. Consider an arbitrary distribution of free classical particles. Apart from shifts in momentum and position, this distribution is identical when viewed from an inertial frame and from a frame subject to an arbitrary acceleration—in particular, an acceleration opposite that produced by our uniform force $\mathbf{F}_{\mathrm{CL}}(t)$.

In addition, if our free distribution is viewed from a frame that is so accelerated, it will be indistinguishable from the same distribution subject to $\mathbf{F}_{\mathrm{CL}}(t)$, but viewed from an inertial frame. We thus find that, when viewed from an inertial frame, the free and uniformly-forced classical distributions must be identical (up to momentum and position shifts).

By analogy, we expect free and uniformly-forced quantum probability distributions to be identical, apart from an appropriate translation, when viewed from an inertial frame. The forced quantum state should include a phase change reflecting the momentum shift. This is precisely what we found in (8), (20) and (23).

Acknowledgment

The author has greatly benefited from many discussions with and suggestions by Professor Ralph Baierlein, whose assistance is hereby gratefully acknowledged.

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